

Question 1

A Bungee Jumper

- (a) The jumper comes to rest when

lost gravitational potential energy = stored strain energy

$$mgy = \frac{1}{2} k (y-L)^2$$

$$ky^2 - 2y(kL + mg) + kL^2 = 0$$

This is solved as a quadratic.

$$\begin{aligned} y &= \frac{2(kL + mg) \pm \sqrt{4(kL + mg)^2 - 4k^2 L^2}}{2k} \\ &= \frac{kL + mg \pm \sqrt{2mgkL + m^2 g^2}}{k} \end{aligned}$$

Need positive root; lower position of rest (other root after initial rise).

- (b) The maximum speed is attained when the acceleration is zero and forces balance;
i.e. when $mg = kx$

Also kinetic energy = lost potential energy – strain energy within elastic rope

$$\frac{1}{2} m v^2 = mg(L + x) - \frac{1}{2} kx^2$$

$$x = \frac{mg}{k}$$

$$v^2 = 2g\left(L + \frac{mg}{k}\right) - \frac{mg^2}{k}$$

$$v = \sqrt{2gL + \frac{mg^2}{k}}$$

- (c) Time to come to rest = time in free fall + time in SHM of rope to stop stretching

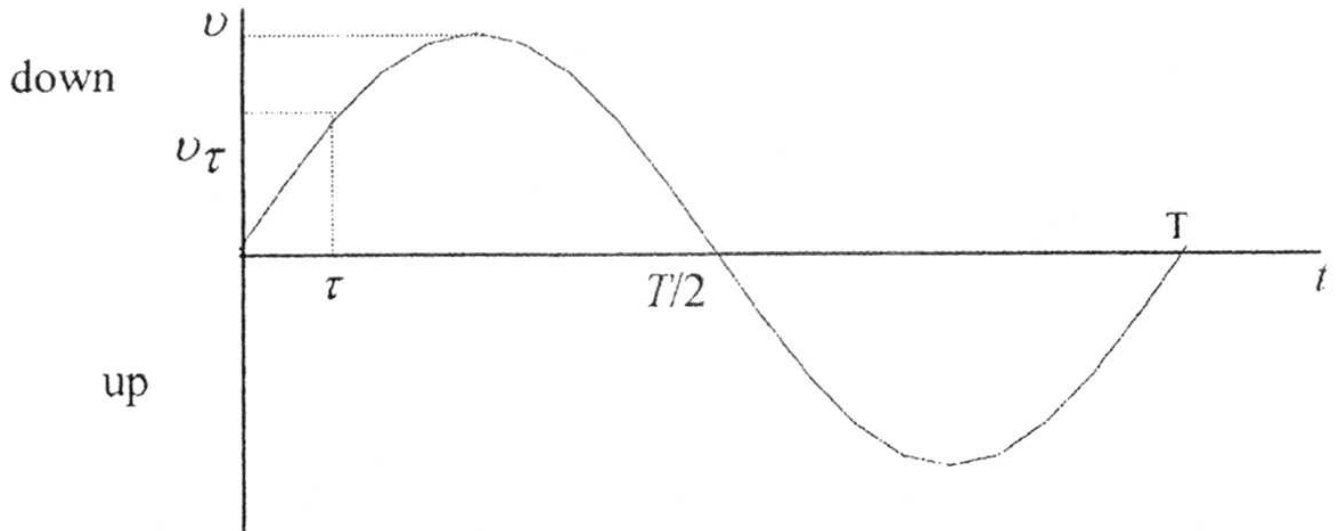
$$\text{Length of free fall} = L = \frac{1}{2} g t_f^2$$

$$\text{Therefore } t_f = \sqrt{\frac{2L}{g}}$$

The jumper enters the SHM with free fall velocity $= gt_f = \sqrt{2gL} = v_\tau$

$$\text{Period of SHM} = 2\pi\sqrt{\frac{m}{k}} = T$$

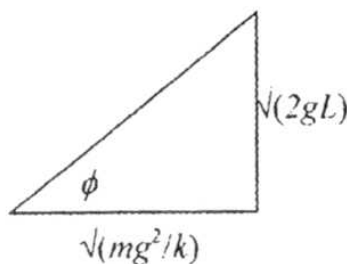
We represent a full SHM cycle by



The jumper enters the SHM at time τ given by

$$\tau = \frac{1}{\omega} \sin^{-1} \frac{v_\tau}{v} = \frac{1}{\omega} \sin^{-1} \frac{\sqrt{2gL}}{v}$$

Jumper comes to rest at one half cycle of the SHM at total time given by
 $= t_f + (T/2 - \tau)$



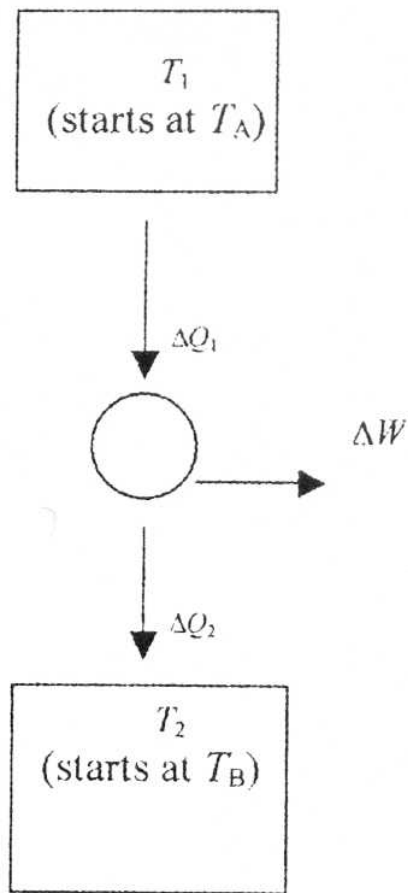
$$\begin{aligned} &= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega} \sin^{-1} \frac{\sqrt{2gL}}{v} \\ &= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega} \sin^{-1} \frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}} \\ &= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}} \left\{ \pi - \sin^{-1} \frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}} \right\} \end{aligned}$$

This is the same as

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}} \left\{ \frac{\pi}{2} + \cos^{-1} \frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}} \right\}$$

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}} \tan^{-1} \left\{ \sqrt{\frac{2kL}{mg}} \right\}$$

B Heat Engine Question



In calculating work obtainable, we assume no loss (friction etc.) in engine working.

$$\Delta Q_1 = \text{energy from body A} \\ = -ms\Delta T_1 \quad (\Delta T_1 - \text{ve})$$

$$\Delta Q_2 = ms\Delta T_2 \quad (\Delta T_2 + \text{ve})$$

(a) **For maximum amount of mechanical energy assume Carnot engine**

$$\frac{\Delta Q_1}{T_1} = \frac{\Delta Q_2}{T_2} \text{ throughout operation (second law)}$$

$$\text{But } \Delta Q_1 = -ms\Delta T_1 \quad \text{and} \quad \Delta Q_2 = ms\Delta T_2$$

$$-ms \int_{T_A}^{T_0} \frac{dT_1}{T_1} = ms \int_{T_B}^{T_0} \frac{dT_2}{T_2}$$

$$\ln \frac{T_A}{T_0} = \ln \frac{T_0}{T_B}$$

$$T_0^2 = T_A T_B$$

$$T_0 = \sqrt{T_A T_B}$$

$$Q_1 = -ms \int_{T_A}^{T_0} dT_1 = ms(T_A - T_0)$$

$$Q_2 = ms \int_{T_B}^{T_0} dT_2 = ms(T_0 - T_B)$$

$$W = Q_1 - Q_2$$

$$W = ms(T_A - T_0 - T_0 + T_B) = ms(T_A + T_B - 2T_0) = ms(T_A + T_B - 2\sqrt{T_A T_B})$$

$$\text{or } ms(\sqrt{T_A} - \sqrt{T_B})^2$$

(d) Numerical example:

Mass = volume $\times$ density

$$\begin{aligned} W &= 2.50 \times 1.00 \times 10^3 \times 4.19 \times 10^3 \times (350 + 300 - 2\sqrt{350 \times 300}) \text{ J} \\ &= 20 \times 10^6 \text{ J} \\ &= 20 \text{ MJ} \end{aligned}$$

C Radioactivity and age of the Earth

(a) $N = N_0 e^{-\lambda t}$ $N_0 = \text{original number}$

$$n = N_0(1 - e^{-\lambda t})$$

Therefore $n = N e^{\lambda t}(1 - e^{-\lambda t}) = N(e^{\lambda t} - 1)$

So $n = N(2^{t/\tau} - 1)$ where τ is half-life

or as $\lambda = \frac{\ln 2}{T} = \frac{0.6931}{T}$, $n = N(e^{\frac{0.6931t}{T}} - 1)$

$^{206}_{\text{Pb}} n = ^{238}_{\text{U}} N(2^{t/4.50} - 1)$ or $^{206}_{\text{Pb}} n = ^{238}_{\text{U}} N(e^{0.1540t} - 1)$ where time t is in 10^9 years

(b) $^{207}_{\text{Pb}} n = ^{235}_{\text{U}} N(2^{t/0.710} - 1)$ or $^{207}_{\text{Pb}} n = ^{235}_{\text{U}} N(e^{0.9762t} - 1)$

(c) In mixed uranium (i.e. containing Pb of both natural and radioactive origin)

$^{204}_{\text{Pb}} : ^{206}_{\text{Pb}} : ^{207}_{\text{Pb}}$ have proportions	1.00 : 29.6 : 22.6
In pure lead (no radioactivity)	1.00 : 17.9 : 15.5

Therefore for radioactively produced lead by subtraction

$^{204}_{\text{Pb}} : ^{206}_{\text{Pb}} : ^{207}_{\text{Pb}}$ have proportions	1.00 : 29.6 : 22.6
In pure lead (no radioactivity)	1.00 : 17.9 : 15.5

Therefore for radioactivity produced lead by subtraction

$^{206}_{\text{Pb}} : ^{207}_{\text{Pb}}$	11.7 : 7.1
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Dividing equations from (a) and (b) gives

$$\frac{^{206}_{\text{Pb}} n}{^{207}_{\text{Pb}} n} = \frac{^{238}_{\text{U}} N}{^{235}_{\text{U}} N} \left\{ \frac{2^{t/4.50} - 1}{2^{t/0.710} - 1} \right\} \text{ or } \frac{^{206}_{\text{Pb}} n}{^{207}_{\text{Pb}} n} = \frac{^{238}_{\text{U}} N}{^{235}_{\text{U}} N} \left\{ \frac{e^{0.1540t} - 1}{e^{0.9762t} - 1} \right\}$$

$$\frac{11.7}{7.1} = 137 \left\{ \frac{2^{t/4.50} - 1}{2^{t/0.710} - 1} \right\} \text{ or } \frac{11.7}{7.1} = 137 \left\{ \frac{e^{0.1540t} - 1}{e^{0.9762t} - 1} \right\}$$

$$0.0120 \{2^{t/0.710} - 1\} = \{2^{t/4.50} - 1\}$$

$$\text{or } 0.0120 \{e^{0.9762t} - 1\} = \{e^{0.1540t} - 1\}$$

(d)

Assume $T \gg 4.50 \times 10^9$ and ignore 1 in both brackets:

$$0.0120 \{2^{T/0.710}\} = \{2^{T/4.50}\} \text{ or } 0.0120 \{e^{0.9762T}\} = \{e^{0.1540T}\}$$

$$0.0120 = \{2^{T/4.50 - T/0.710}\} = 2^{T(0.222 - 1.4084)} = 2^{-1.1862T}$$

$$T = -\frac{\log 0.0120}{\log 2 \times 1.1862} = 5.38$$

$$T = 5.38 \times 10^9 \text{ years}$$

$$\text{or } 0.0120 = e^{-0.8222T} \quad T = \frac{\ln 0.0120}{-0.8222} = \frac{-4.4228}{-0.8222} = 5.38$$
$$T = 5.38 \times 10^9 \text{ years}$$

(e) T is not $\gg 4.50 \times 10^9$ years but is $> 0.71 \times 10^9$ years

We can insert the approximate value for T (call it $T^* = 5.38 \times 10^9$ years) in the $2^{T/4.50}$ term and obtain a better value by iteration in the rapidly changing $2^{T/0.710}$ term). We now leave in the -1's, although the -1 on the right-hand side has little effect and may be omitted).

$$\text{Either} \quad 0.0120(2^{T/0.710} - 1) = 2^{T^*/4.50} - 1$$
$$2^{T/0.710} - 1 = \frac{2^{1.1956} - 1}{0.0120} = \frac{2.2904 - 1}{0.0120} = 107.5$$
$$T = 0.710 \frac{\log 108.5}{\log 2} = 4.80(0)$$

Put $T^* = 4.80(0) \times 10^9$ years

$$2^{T/0.710} = \frac{2^{1.0668} - 1}{0.0120} = \frac{2.0948 - 1}{0.0120} = 91.2$$
$$T = 0.710 \frac{\log 91.2}{\log 2} = 4.62(3)$$

Further iteration gives 4.52

So more accurate answer for T to be in range 4.6×10^9 years to 4.5×10^9 years (either acceptable).

D Spherical charge

(a) Charge density = $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$ within sphere

$x \leq R$ Field at distance x :

$$E = \frac{\frac{4}{3}\pi x^3 \rho}{4\pi\epsilon_0 x^2} = \frac{Qx}{4\pi\epsilon_0 R^3}$$

$x > R$ Field at distance x from the centre: $E = \frac{Q}{4\pi\epsilon_0 x^2}$

(b) Method 1

Energy density is $\frac{1}{2}\epsilon_0 E^2$.

$x \leq R$

Energy in a thin shell of thickness δx at radius x is given by

$$= \frac{1}{2}\epsilon_0 E^2 4\pi x^2 \delta x = \frac{1}{2}4\pi\epsilon_0 \frac{Q^2 x^2}{(4\pi\epsilon_0)^2 R^6} x^2 \delta x$$

$$\text{Energy within the spherical volume} = \frac{1}{2} \frac{Q^2}{(4\pi\epsilon_0)R^6} \int_{x=0}^{x=R} x^4 dx = \frac{1}{40} \frac{Q^2}{\pi\epsilon_0} \frac{1}{R}$$

$x > R$

$$\text{Energy within spherical shell} = \frac{1}{2}\epsilon_0 E^2 4\pi x^2 \delta x = \frac{1}{2}4\pi\epsilon_0 \frac{Q^2}{(4\pi\epsilon_0)^2 x^4} x^2 \delta x$$

Energy within the spherical volume for $x > R$

$$= \frac{1}{2} \frac{Q^2}{(4\pi\epsilon_0)} \int_{x=R}^{x=\infty} \frac{1}{x^2} dx = \frac{1}{8} \frac{Q^2}{\pi\epsilon_0} \frac{1}{R}$$

$$\text{Total energy associated with the charge distribution} = \frac{1}{40} \frac{Q^2}{\pi\epsilon_0} \frac{1}{R}$$

$$+ \frac{1}{8} \frac{Q^2}{\pi\epsilon_0} \frac{1}{R}$$

$$= \frac{3}{20} \frac{Q^2}{\pi\epsilon_0} \frac{1}{R}$$

Method 2

A shell with charge $4\pi x^2 \delta x \rho$ moves from ∞ to the surface of a sphere radius x

where the electric potential is

$$\frac{\frac{4}{3}\pi x^3 \rho}{4\pi \epsilon_0 x} = \frac{x^2 \rho}{3\epsilon_0}$$

and will therefore gain electrical potential energy $\left(\frac{x^2 \rho}{3\epsilon_0}\right)(4\pi x^2 \delta x)$

$$\text{Total energy of complete sphere} = \int_{x=0}^{x=R} \frac{4\pi \rho^2 x^4}{3\epsilon_0} dx = \frac{4}{15} \frac{\pi \rho^2 R^5}{\epsilon_0}$$

$$\text{Putting } Q = \text{charge on sphere} = \frac{4}{3}\pi R^3 \rho, \quad \rho = \frac{3Q}{4\pi R^3}$$

$$\text{So that total energy is} = \frac{4}{15} \pi \left(\frac{9Q^2}{16\pi^2 R^6} \right) \frac{R^5}{\epsilon} = \frac{3}{20} \frac{Q^2}{\pi \epsilon_0 R}$$

$$(c) \quad \text{Binding energy } E_{\text{binding}} = E_{\text{electric}} - E_{\text{nuclear}}$$

Binding energy is a negative energy

$$\text{Therefore } -8.768 = E_{\text{electric}} - 10.980 \text{ MeV per nucleon}$$

$$E_{\text{electric}} = 2.212 \text{ MeV per nucleon}$$

$$\text{Radius of cobalt nucleus is given by } R = \frac{3}{20} \frac{Q^2}{\pi \epsilon_0 E_{\text{electric}}^{\text{total}}}$$

$$= \frac{3 \times 27^2 \times (1.60 \times 10^{-19})^2}{20 \times \pi \times 8.85 \times 10^{-12} \times 2.212 \times 10^6 \times 57 \times 1.60 \times 10^{-19}} \text{ m}$$

$$= 5.0 \times 10^{-15} \text{ m}$$

$$T = \frac{4\rho d \ln 2}{B^2} = \frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^3 \times 0.6931}{(44.5 \times 10^{-6} \times 0.4384)^2} \text{ s}$$

$$= 1.10(2) \times 10^6 \text{ s } (=306 \text{ hr } = 12 \text{ days } 18 \text{ hr})$$

Method 1 Equating energyHorizontal component of magnetic field B inducing emf in ring:

$$B = 44.5 \times 10^{-6} \cos 64^\circ$$

Magnetic flux through ring at angle $\theta = B\pi a^2 \sin \theta$ where a = radius of ring

$$\begin{aligned} \text{Instantaneous emf} &= \frac{d\phi}{dt} = B\pi a^2 \frac{d \sin \omega t}{dt} \quad \text{where } \omega = \text{angular velocity} \\ &= B\pi a^2 \omega \cos \omega t = B\pi a^2 \omega \cos \theta \end{aligned}$$

$$\text{R.m.s. emf over 1 revolution} = \frac{B\pi a^2 \omega}{\sqrt{2}}$$

$$\text{Average resistive heating of ring} = \frac{B^2 \pi^2 a^4 \omega^2}{2R}$$

$$\text{Moment of inertia} = \frac{1}{2} m a^2$$

$$\text{Rotational energy} = \frac{1}{4} m a^2 \omega^2 \quad \text{where } m = \text{mass of ring}$$

$$\text{Power producing change in } \omega = \frac{d}{dt} \left\{ \frac{1}{4} m a^2 \omega^2 \right\} =$$

$$\frac{1}{4} m a^2 2\omega \frac{d\omega}{dt}$$

$$\text{Equating:} \quad \frac{1}{2} m a^2 \omega \frac{d\omega}{dt} = - \frac{B^2 \pi^2 a^4 \omega^2}{2R}$$

$$\frac{d\omega}{\omega} = - \frac{B^2 \pi^2 a^2}{mR} dt$$

If T is time for angular velocity to halve,

$$\int_{\omega}^{\omega/2} \frac{d\omega}{\omega} = - \int_0^T \frac{B^2 \pi^2 a^2}{mR} dt$$

$$\ln 2 = \frac{B^2 \pi^2 a^2}{mR} T$$

But $R = \frac{2\pi a \rho}{A}$ where A is cross-sectional area of copper ring $m = 2\pi a d A$ (d = density)

$$\ln 2 = \frac{B^2 \pi^2 a^2 T}{\frac{2\pi a \rho}{A} 2\pi a d A} = \frac{B^2 T}{4\rho d}$$

$$\begin{aligned} T &= \frac{4\rho d \ln 2}{B^2} = \frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^3 \times 0.6931}{(44.5 \times 10^{-6} \times 0.4384)^2} \text{ s} \\ &= 1.10(2) \times 10^6 \text{ s} \quad (= 306 \text{ hr} = 12 \text{ days } 18 \text{ hr}) \end{aligned}$$

Method 2 Back Torque

Horizontal component of magnetic field $= B = 44.5 \times 10^{-6} \cos 64^\circ$

Cross-section of area of ring is A

Radius of ring $= a$

Density of ring $= d$

Resistivity $= \rho$

ω = angular velocity (ω positive when clockwise)

$$\text{Resistance } R = \rho \frac{2\pi a}{A}$$

$$\text{Mass of ring } m = 2\pi a A d$$

$$\text{Moment of inertia } = M = \frac{1}{2} m a^2$$

Magnetic flux through ring at angle $\theta = B\pi a^2 \sin \theta$

$$\text{Instantaneous emf} = \frac{d\phi}{dt} = B\pi a^2 \frac{d \sin \omega t}{dt} = B\pi a^2 \omega \cos \omega t = B\pi a^2 \omega \cos \theta$$

$$\text{Induced current} = I = B\pi a^2 \omega \cos \theta / R$$

$$\text{Torque opposing motion} = (B\pi a^2 \cos \theta) I = \frac{1}{R} (B\pi a^2)^2 \omega \cos^2 \theta$$

$$\text{Work done in small } \delta\theta = \frac{1}{R} (B\pi a^2)^2 \omega \frac{1}{2} (\cos 2\theta + 1) \delta\theta$$

Average torque = (work done in 2π revolution) / 2π

$$= \frac{1}{2\pi R} (B\pi a^2)^2 \omega \frac{1}{2} 2\pi = \frac{1}{2R} (B\pi a^2)^2 \omega$$

$$\text{This equals } M \frac{d\omega}{dt} \text{ so that } M \frac{d\omega}{dt} = - \frac{B(\pi a^2) B(\pi a^2)}{(\rho / A)(2\pi a)} \frac{1}{2} \omega$$

$$\frac{1}{2} (2\pi a A d) a^2 \frac{d\omega}{dt} = - \frac{B^2 (\pi a^2)^2 A}{4 \rho \pi a} \omega$$

$$\frac{d\omega}{dt} = - \frac{B^2}{4 \rho d} \omega$$

$$\int_{\omega}^{\omega/2} \frac{d\omega}{\omega} = \int_0^T \frac{B^2}{4 \rho d} dt$$

$$\ln 2 = \frac{B^2 T}{4 \rho d}$$

$$T = \frac{4 \rho d \ln 2}{B^2} = \frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^3 \times 0.6931}{(44.5 \times 10^{-6} \times 0.4384)^2} \text{ s}$$

$$= 1.10(2) \times 10^6 \text{ s} = 306 \text{ hr} = 12 \text{ days } 18 \text{ hr}$$