General Instructions: Theoretical Examination (30 points)

May 8, 2025

The theoretical examination lasts for 5 hours, consists of 3 questions and is worth a total of 30 point, 10 points each.

Before the exam

- You must not open the envelope containing the problems before the sound signal indicating the beginning of the examination.
- The beginning and end of the examination will be indicated by a sound signal. There will be announcements every hour indicating the elapsed time, as well as fifteen minutes before the end of the examination (before the final sound signal).

During the exam

- Dedicated answer sheets are provided for writing your answers. Enter the final answers into the appropriate boxes in the corresponding answer sheet (marked **A**). For every problem, there are extra blank work sheets for carrying out detailed work (marked **W**). Always use the work sheets that belong to the problem you are currently working on (check the problem number in the header). If you have written something on any sheet which you do not want to be graded, cross it out. Only use the front side of every page.
- In your answers, try to be as concise as possible: use equations, logical operators and sketches to illustrate your thoughts whenever possible. Avoid the use of long sentences.
- Only use **pen** provided by the organizer in your answer sheet
- Please give an appropriate number of significant digits when stating numbers.
- You may often be able to solve later parts of a problem without having solved the previous ones.
- You are not allowed to leave your working place without permission. If you need any assistance (need to refill your drinking water bottle, broken calculator, need to visit a restroom, etc), please take three steps back and draw the attention of a team guide.

At the end of the exam

- At the end of the examination you must stop writing immediately.
- Sort the corresponding sheets in the following order: cover sheet (**C**), questions (**Q**), answer sheets (**A**), work sheets (**W**).
- Put all the sheets of a single task in the corresponding envelope. Also put the three small envelopes and the general instructions (**G**) into the big envelope. Also hand in empty sheets. You are not allowed to take any sheets of paper out of the examination area.
- Please take the calculator provided by the organizer with you.
- Please take the writing equipment with you.
- Wait at your table until your envelope is collected. Once the envelope is collected, your guide will escort you out of the examination area.

Precession of the Earth's axis (10.0 points)

Introduction

It has been known since ancient times that the Earth's axis of rotation precesses. That is, the axis itself rotates around the line perpendicular to the ecliptic plane, i.e., the plane containing the Earth's orbit around the Sun. Ancient Greek astronomer *Hipparchus* concluded that the annual angular displacement of the axis was approximately 45'' (seconds of arc), which would imply that the period of axial precession is around 29000 years. Modern measurements indicate that the period is approximately 25800 years. In this problem, you are asked to investigate this phenomenon using Newtonian mechanics.

You may need the following constants:

- gravitational constant: $G = 6.67 \times 10^{-11}$ Nm^2/kg^2
- average radius of Earth: $R = 6.371 \times 10^6$ m
- mass of the Earth: $M_E = 5.972 \times 10^{24}$ kg
- average distance of the Sun from the Earth: $d_{SE} = 1.496 \times 10^{11}$ m
- mass of the Sun: $M_S = 1.989 \times 10^{30}$ kg
- average distance of the Moon from the Earth: $d_{ME} = 3.844 \times 10^8$ m
- mass of the Moon: $M_M = 7.348 \times 10^{22}$ kg
- Earth's axial tilt: $\alpha = 23.5^\circ$

Part A. The shape of the Earth (1.0 points)

The Sun and the Moon exert nonzero torques on the Earth because of its non-spherical shape, giving rise to its axial precession. The main reason behind the Earth's non-spherical shape is the centrifugal force caused by the Earth's rotation about its axis. The tectonic plates located on the Earth's surface have deformed over millions of years to minimize stress within them. Therefore, as an approximation, let us model the Earth as a large liquid droplet of uniform density whose shape is determined by centrifugal and gravitational forces. In this model, the Earth's surface is an oblate spheroid (ellipsoid of revolution) characterized by the polar radius R_p and the equatorial radius R_e (see *Figure A.1*).

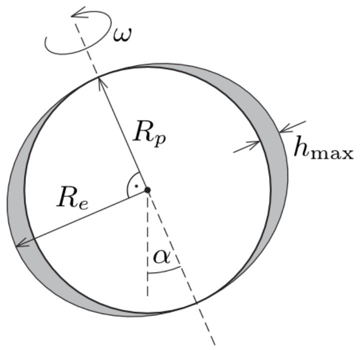

Figure A.1. The ellipsoidal shape of the Earth. The polar and equatorial radii are indicated. $\alpha = 23.5^\circ$ is the angle between the Earth's axis of rotation and the normal of the ecliptic plane.

The difference between the equatorial and polar radii of the Earth, $h_{\text{max}} = R_e - R_p$ is much smaller than the average radius $R = (R_e + R_p)/2$. Up to a dimensionless factor, the value of h_{max} can be expressed in terms of the angular speed of the Earth's rotation ω, its mass M_E and average radius R as

$$h_{\text{max}} \propto G^{-1}\omega^{\beta}M_E^{\gamma}R^{\delta},$$

where G is the gravitational constant, and β, γ and δ are constant exponents.

A.1	Find the values of exponents β, γ and δ.	0.8 pt

A.2	Calculate the numerical value of h_{max} assuming that the dimensionless factor in the relation given above equals 1.	0.2 pt

Regardless of whether you were able to find h_{max} in part **A.2.**, use the empirical value $h_{\text{max}} = 21$ km in the following questions.

Part B. The time-averaged gravitational field of the Sun (3.2 points)

To see why the Sun exerts a nonzero torque (with respect to the center of the Earth) on our planet, consider *Figure B.1* below. The difference in distance from the Sun causes the gravitational force F_1 to be greater than its counterpart F_2.

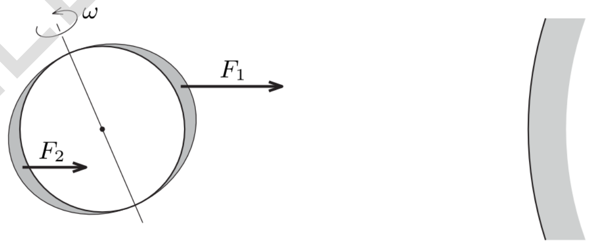

Figure B.1. Explanation of the nonzero torques exerted by the Sun (right side of the figure) on the Earth (left side).

The magnitude of this torque acting on the Earth varies continuously during the year. In the position shown in *Figure B.1*, the torque is maximal, a quarter of a year later, due to symmetry, the torque becomes zero. After half a year, it reaches the maximum again, three-quarters of a year later it is zero once again, and so on. Since the period of axial precession is much larger than one year, this time-dependent torque can be approximated well by its one-year average.

To calculate the average torque exerted by the Sun on the Earth, let us determine first the time average of the gravitational field generated by the Sun in the vicinity of the Earth. This average can be calculated as the field of a uniformly dense mass ring, a *Sun ring*, whose mass equals the mass of the Sun M_S and whose radius equals the average distance between the Sun and the Earth d_{SE} (see *Figure B.2*).

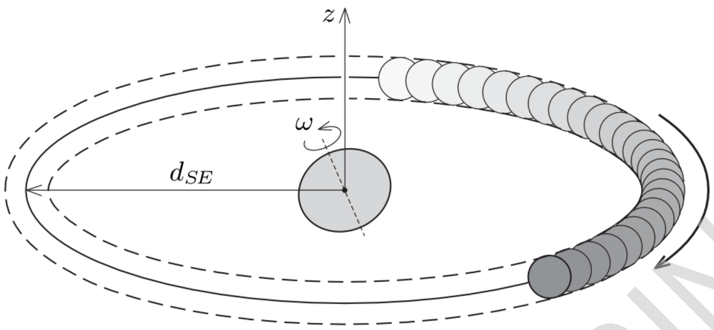

Figure B.2. Time averaging is equivalent to uniformly spreading the Sun along the circle of radius d_{SE}.

Let our cylindrical coordinate system have the origin at the center of the Earth, and let the z axis be perpendicular to the ecliptic plane (i.e. the plane of the ring). The axis of rotation of the Earth makes an angle of $\alpha = 23.5^\circ$ with the z axis.

B.1	Find the direction and magnitude of the gravitational field generated by the Sun ring at a point on the z axis. Write your answer in terms of M_S, d_{SE}, and the coordinate z. Assume that $\|z\| \ll d_{SE}$.	1.0 pt

B.2	Find the direction and magnitude of the gravitational field generated by the Sun ring at a point in the ecliptic plane whose distance from the origin is r. Assume that $r \ll d_{SE}$.	2.2 pt

Part C. The torque acting on the Earth (2.6 points)

In this section, you are asked to determine the torque exerted on the Earth due to the gravitational field obtained in **Part B**. For simplicity, consider the Earth as a rigid body with homogeneous mass distribution. Let us take into account that the rotational ellipsoid can be imagined as if we removed excess parts from a sphere with the equatorial radius of Earth R_e (see *Figure C.1*).

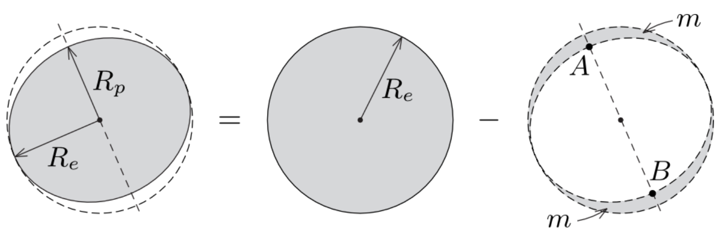

Figure C.1. The ellipsoidal shape of the Earth can be imagined as if the excess parts were removed from a complete sphere of radius R_e.

C.1	Find the mass m of one of the two excess regions indicated in *Figure C.1*. Express your answer in terms of h_{max}, the mass of the Earth M_E, and its polar radius R_p.	0.8 pt

It can be shown that the torque acting on the excess regions is equivalent to the torque acting on two point masses, each with a mass equal to $2m/5$, positioned at the endpoints A and B of the polar diameter (see *Figure C.1*).

C.2	Given this idea, find the torque τ exerted by the Sun ring on the Earth. Express your answer in terms of M_E, M_S, d_{SE}, R (the average radius), h_{max} and the angle α. You can use that $h_{\text{max}} \ll R$.	1.8 pt

Part D. Angular speed of the precession of the Earth's axis (2.0 points)

The Earth's axis of rotation moves *very slowly* around the z axis in a conical motion. That is, it precesses.

D.1	Give an expression for the period T_1 of precession of the Earth's axis. Express your answer in terms of M_S, d_{SE}, the angular speed ω of the Earth's rotation, h_{max}, R and α.	1.8 pt

D.2	Calculate the precession period T_1 in years.	0.2 pt

Part E. The effect of the Moon (1.2 points)

The value obtained in **Part D** is much larger than the observed value. The reason for this is that so far we have only considered the torque exerted by the Sun, and neglected the effect of the Moon. In the following calculations, assume that the Moon's orbit is in the ecliptic plane, and that the orbit of the Moon around the Earth is a circle of radius d_{ME}. Let us denote the mass of the Moon by M_M and the period of precession in this modified model by T_2.

E.1	By what factor T_2/T_1 does the period of precession of the Earth's axis change if we also take into account the torque exerted by the Moon? Give your answer in terms of d_{ME}, d_{SE}, M_S and M_M.	1.0 pt

E.2	By substituting the data, calculate the period of precession T_2 in years.	0.2 pt

Waves and Phase Transitions in Spin Systems (10.0 points)

Introduction

In classical physics, angular momentum arises from the motion of an object around an axis - whether it be a spinning top, a rotating planet, or an orbiting electron in the atom. However, in quantum physics, fundamental particles possess an intrinsic and quantized form of angular momentum called *spin*. This property plays a crucial role in various physical phenomena, ranging from materials properties, such as magnetism, to modern applications, such as quantum computing.

In this problem we will treat spin classically, which will lead to some qualitatively correct results. You will explore the physics of spin systems through spin-spin interactions, evolution under magnetic fields, and statistical physics to understand the emergence of spin waves and phase transitions in magnets.

Useful information:

$\cosh(x) \equiv \frac{e^x+e^{-x}}{2}$, $\sinh(x) \equiv \frac{e^x-e^{-x}}{2}$, $\tanh(x) \equiv \frac{\sinh(x)}{\cosh(x)} \approx x - \frac{1}{3}x^3$ for $|x| \ll 1$

The magnetic field due to a magnetic dipole of moment $\vec{\mu}$ at a position $\vec{r}$ away from it is given by (μ_0 is is the vacuum permeability):

$$\vec{B} = \frac{\mu_o}{4\pi}\left(\frac{3(\vec{\mu}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{\mu}}{r^3}\right) \tag{1}$$

Part A. Precession and interactions of magnetic dipoles (1.2 points)

Consider a ring of radius R, total mass M, and charge $Q > 0$ distributed uniformly. The ring rotates with an angular speed ω around a perpendicular axis that passes through its center of mass.

A.1	It is possible to write the ring's magnetic moment $\vec{\mu}$ in terms of its angular momentum $\vec{L}$ as $\vec{\mu} = \gamma\vec{L}$. Find the constant γ, called the *gyromagnetic ratio*, of this system in terms of Q and M.	0.3 pt

The ring is placed in a weak uniform magnetic field $\vec{B} = B\hat{z}$, making an angle θ with $\vec{\omega}$, see *Figure A.1*.

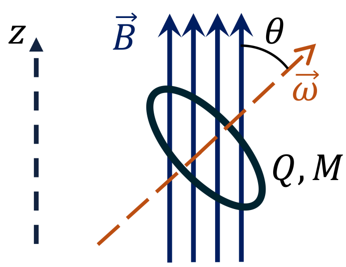

Figure A.1.

A.2	Find the angular frequency ω_L of the angular momentum precession (the so-called Larmor frequency) due to the external magnetic field in terms of B and γ. Take the positive direction to be counter-clockwise with respect to $+z$.	0.4 pt

Now we turn off the external magnetic field and place an identical ring at a horizontal distance $d \gg R$ from the original ring such that the magnetic moment of the new ring $\vec{\mu}_2$ makes an angle θ with $\vec{\mu}_1$, see *Figure A.2*.

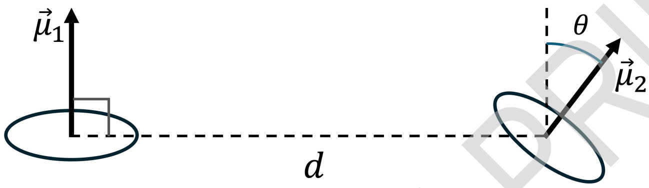

Figure A.2.

A.3	The magnetic interaction energy between the two rings can be written as $U = J_0 \vec{L}_1 \cdot \vec{L}_2$, where J_0 is a constant and $\vec{L}_i$ is the angular momentum of the ith ring. Find J_0 in terms of γ, d and fundamental constants.	0.5 pt

Part B. Spin Waves (4.5 points)

In what follows we investigate the dynamics of spins. A spin is a particle with intrinsic angular momentum $\vec{S}$, which has an associated magnetic moment $\vec{\mu}$ related to $\vec{S}$ via the gyromagnetic ratio as in **Part A.1**, $\vec{\mu} = \gamma \vec{S}$.

The magnetic dipoles of two spins interact with each other. However, this interaction is negligible compared to another interaction arising from a quantum mechanical origin, which is not present in classical systems. Interestingly, the energy associated with this quantum interaction has the same form which we found in **Part A.3**, scaling with $\vec{S}_1 \cdot \vec{S}_2$, albeit with the opposite sign.

Now we will look at a very long chain of spins. The positions of the spins are fixed along the x-axis, with a distance a separating them, see *Figure B.1*. We will approximate the total energy of the system by considering the interactions between nearest neighbors only, so that the energy can be written as

$$E = -J \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

where $J > 0$ is the interaction strength, and $\vec{S}_i$ is the spin angular momentum vector of the ith dipole, with magnitude S. The spin vectors are free to rotate in three dimensions. Notice that the sign of the energy is different from the last part. This interaction is purely quantum mechanical.

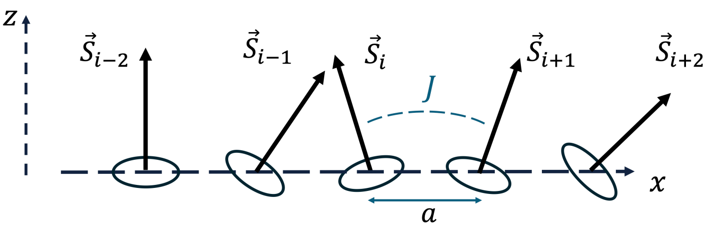

Figure B.1.

B.1	The energy terms containing $\vec{S}_i$ in the sum above can be viewed as the interaction energy between an effective magnetic field $\vec{B}_{i,\text{eff}}$ and the magnetic moment of $\vec{S}_i$. Find $\vec{B}_{i,\text{eff}}$ and express your answer in terms of J, the gyromagnetic ratio γ, and other spins $\vec{S}_j$ (specify the indices j in relation to i)	0.3 pt

B.2	Using the concept of effective magnetic field, express the rate of change of the ith spin vector, $d\vec{S}_i/dt$, in terms of $J, \vec{S}_i$, and other spins $\vec{S}_j$ (specify the indices j in relation to i).	0.3 pt

For the rest of **Part B**, assume that the system is highly magnetized along the z direction, so we can use the approximations $S_{i,z} \approx S$ and $dS_{i,z}/dt \approx 0$ for each spin, see *Figure B.2*. In this regime, the set of equations describing the spins time evolution is satisfied by a traveling wave solution for $S_{i,x}$ and $S_{i,y}$ characterized by a wave vector k and angular frequency ω.

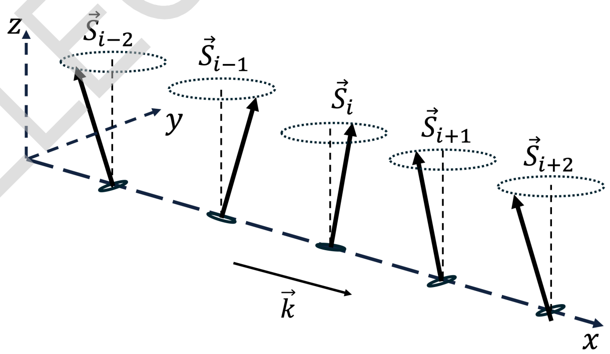

Figure B.2.

B.3	Find the relationship between ω and k (known as the dispersion relation, $\omega(k)$) for the spin waves in terms of J, S and a. *Hint*: express the position of the ith spin as $x = a \cdot i$.	2.0 pt

The spin wave described above carries energy and momentum. At low energies, the relation between its energy and momentum resembles that of a massive classical particle with an effective mass m_{eff}, a concept known as a *quasi-particle*.

B.4	For small k ($k \ll 1/a$), find the effective mass m_{eff} of the spin wave. Express your answer in terms of J, S, a and fundamental constants.	0.6 pt

Spin waves can be experimentally probed using inelastic neutron scattering. Although neutrons have zero net charge, they have a finite spin, allowing them to interact with other spins.

B.5	Suppose that initially, all the spins in the chain are pointing along the z direction. A neutron with low energy travels on the $x-y$ plane making an incident angle θ_{in} with the chain and scatters with an angle θ_{out} as shown in *Figure B.3*. Assuming the neutron excites a single low wave vector spin wave, find the effective mass m_{eff} of the spin wave, in terms of $\theta_{\text{in}}, \theta_{\text{out}}$ and the neutron mass m_n. Assume that the chain stays at rest.	1.3 pt

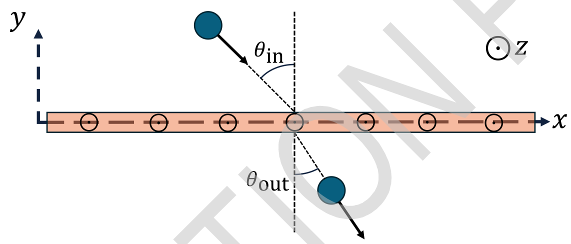

Figure B.3.

Part C. Phase transitions in spin chains (4.3 points)

Next we consider the same chain made of N spins from **Part B**, except the spin vectors are now restricted to point either up or down along the z-axis, so that the spin component along z can be written as $S_{i,z} = s_i S$, where $s_i = \pm 1$, see *Figure C.1*. In addition to the nearest neighbor interactions, we could have an external magnetic field pointing along the z-axis so that the total energy of the system is given by

$$E = -\tilde{J}\sum_i s_i s_{i+1} - h\sum_i s_i.$$

We assume $\tilde{J} \geq 0$, and h is a constant dependent on the magnetic field. The spin system is at equilibrium with a heat bath at temperature T. Ignore the edges of the chain.

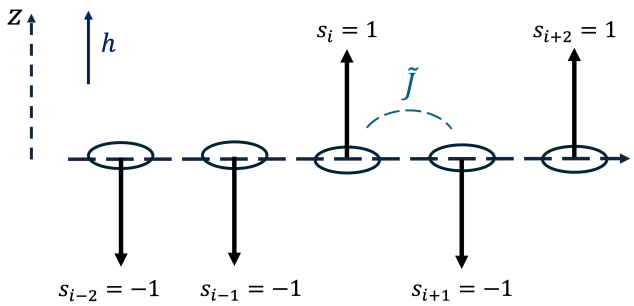

Figure C.1.

C.1	Assume first that $\tilde{J} = 0$, what is the ratio between the probability to find an arbitrary spin aligned to the magnetic field $p_\uparrow$ to being anti-aligned to the magnetic field $p_\downarrow$? Express $p_\uparrow/p_\downarrow$ in terms of h, T and fundamental constants.	0.5 pt

C.2	Find the average polarization of the system $\bar{s} \equiv \frac{1}{N}\sum_i s_i$ for $N \gg 1$ in terms of h, T and fundamental constants. If the magnetic field h can range from $-h_0$ to h_0, make a sketch of $\bar{s}$ as a function of h for the cases $h_o \gg k_B T$, $h_o \approx k_B T$ and $h_o \ll k_B T$.	1.0 pt

In the remaining questions, we turn off the magnetic field, so $h = 0$, and set $\tilde{J} > 0$.

C.3	What is the energy E_g of the ground state (the lowest energy state)? Express your answer in terms of $\tilde{J}$ and N.	0.2 pt

Instead of considering the interactions between each spin and its neighbors, we assume that each spin sees an average polarization $\bar{s}$ from its nearest-neighbors.

C.4	Approximate the energy of the system as a sum over all spins $$E = -\tilde{J}_{\text{eff}} \sum_i s_i$$ and express $\tilde{J}_{\text{eff}}$ in terms of $\tilde{J}$ and $\bar{s}$.	0.2 pt

C.5	Using your result from **C.2**, find an equation that the average polarization $\bar{s}$ must satisfy. The number of solutions to this equation depends on T. Find the critical temperature T_c at which the number of solutions changes. Express your answer in terms of $\tilde{J}$ and fundamental constants.	1.2 pt

C.6	Find all possible values of $\bar{s}$ when $T < T_c$ and $T_c - T \ll T_c$. Express your answers in terms of T and T_c. Sketch all possible values of $\bar{s}$ for the temperature T in the range $0 \leq T \leq 2T_c$.	1.0 pt

C.7	What magnetic phase of matter does $T > T_c$ correspond to? How about when $T < T_c$? Choose between paramagnetic or ferromagnetic.	0.2 pt

Atmospheric Physics (10.0 points)

The Earth's atmosphere is a complex physical system, and predicting its behavior is crucial for environmental and meteorological purposes. However, even the best theoretical models run on modern computers are insufficient to make precise predictions. In this problem, we will attempt to understand some of the basic atmospheric phenomena based on simple models. You might need the following constants: the mean solar power per unit area at Earth, the total solar irradiance $F_s = 1370\ \mathrm{W/m^2}$, molar mass of water $\mu_{\mathrm{H_2O}} \approx 18\ \mathrm{g/mol}$ and average molar mass of air $\mu_{\mathrm{air}} \approx 29\ \mathrm{g/mol}$, The Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\ \mathrm{W/(m^2K^4)}$. All gases in this problem can be treated as ideal gases. Assume that all air molecules have 5 degrees of freedom. You may need the following integral:

$$\int_{-\infty}^{\infty} e^{-ax^2/2}\, dx = \sqrt{\frac{2\pi}{a}}, \quad a > 0.$$

Part A. Surface Temperature of the Earth (1.2 points)

In this section, we study the effect of the atmosphere on the Earth surface's temperature. Assume that Earth and its atmosphere have an albedo $a = 0.3$ for solar radiation, which is the reflected fraction of the total incident radiation. You may use this value in all parts of this problem. In addition, assume the Earth radiates as a black body.

A.1	Express the average net solar power received by the Earth and atmosphere system P_0 in terms of F_s, a and R_E, the radius of the Earth.	0.2 pt

A.2	Estimate the temperature of the Earth's surface T_{g0} assuming that it is at a steady state. Ignore the atmosphere.	0.3 pt

Your answer for **A.2** should be lower than what you would expect. We now consider adding a thin atmospheric layer at temperature T_a, see *Figure A.1*. The atmospheric layer transmits a net fraction t_{sw} of the incident solar radiation and a net fraction t_{lw} of the Earth's thermal radiation. Otherwise, you may treat the atmosphere as a black body.

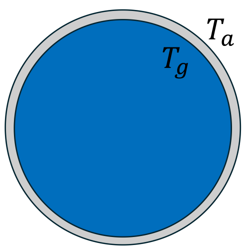

Figure A.1

A.3	Assuming the system is in a steady state, calculate T_g, the temperature of the ground. Use $t_{\text{sw}} = 0.9$ and $t_{\text{lw}} = 0.2$.	0.7 pt

Part B. The absorption spectrum of atmospheric gases (1.8 points)

The infrared radiation emitted by Earth has low energy, incapable of exciting electrons within the molecules, but it has the ability to excite the vibrational and rotational modes of the molecules.

B.1	Consider a simple diatomic molecule modeled as two point masses m_A and m_B connected by a spring with spring constant k. What is the angular frequency of vibrations ω_d?	0.5 pt

B.2	Quantum mechanics dictates that vibrational excitations due to absorbing a photon can only raise the quantum energy level by one. What is the energy of the photon E_p that can excite the vibration in **B.1**? Neglect recoil effects.	0.2 pt

Quantum mechanics forbids the vibrational modes of symmetric diatomic molecules, such as nitrogen and oxygen (the most abundant gasses in the Earth's atmosphere) to be excited by light. This explains why N_2 and O_2 do not contribute to the green house effect. In general, the absorption of light by molecules is governed by the allowed energy transitions in them. However, the energy of the light absorbed does not have to exactly match the energy gap in the molecule. Suppose that a molecule at rest has a spectral line (an allowed transition) at frequency f_0.

B.3	What is the shift in the spectral line $f - f_0$ if the molecule is moving with velocity v towards the emitter such that $\lvert v \rvert \ll c$, where c is the speed of light.	0.2 pt

For a gas at temperature T, the velocity of its molecules is distributed according to Maxwell's distribution. For a molecule of mass m, the probability to find a molecule's velocity along one dimension to be between v and $v + dv$ is $p_1(v)dv$, where $p_1(v)$ is a probability distribution function given by

$$p_1(v) = C \exp\left(-\frac{mv^2}{2k_B T}\right)$$

C is a normalization constant ensuring the probabilities add up to one, and k_B is the Boltzmann constant.

B.4	Find the normalization constant C, assuming that the velocity v could range from $-\infty$ to ∞.	0.2 pt

B.5	Find the probability distribution function $p_2(f)$ to find a molecule with a spectral line f_0 shifted to f due to thermal motion, up to a normalization factor, in terms of f, f_0, T, m and fundamental constants.	0.3 pt

B.6	Sketch $p_2(f)$ as a function of $f - f_0$, and determine the shift $f^\star - f_0$ at which $p_2(f^\star)$ is a fraction $1/e$ of its peak value, where e is the natural number.	0.4 pt

Part C. Stability of air in the atmosphere (2.7 points)

Consider a small cylindrical mass of air at height z above the ground. The pressure and mass density of air at that height are $p(z)$ and $\rho(z)$, respectively, see *Figure C.1*. Assume a uniform downward gravitational field g and that the pressure on the Earth's surface is p_o.

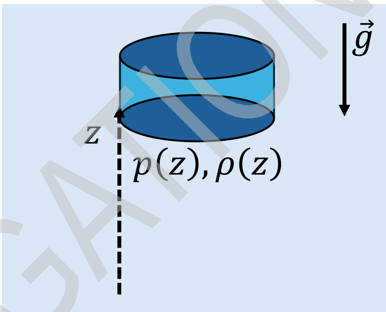

Figure C.1

C.1	Assuming that the small air mass is at hydrostatic equilibrium, derive an expression of the rate of change of pressure with respect to height, dp/dz in terms of g and $\rho(z)$.	0.3 pt

C.2	Express dp/dz in terms of $\mu_{\text{air}}, g, p(z)$ and $T(z)$, the temperature at height z and fundamental constants.	0.2 pt

C.3	Assuming an isothermal atmosphere, $T(z) = T$, find an expression for $p(z)$ in terms of $z, \mu_{\text{air}}, g, p_o, T$ and fundamental constants.	0.2 pt

In a real atmosphere, the temperature is not constant but changes with height. The rate of decrease of temperature with height $\Gamma(z) = -dT/dz$ is called the lapse rate. Consider a small mass of air rising adiabatically in the atmosphere such that it remains at mechanical equilibrium with its surrounding.

C.4	For the adiabatically rising air mass, find the adiabatic lapse rate Γ_a in terms of c_p, the molar specific heat at constant pressure, μ_{air} and g.	0.6 pt

To analyze the stability of an atmosphere, we imagine starting from an equilibrium state, and then perturbing a small mass of air and analyze its response. Consider a small air mass initially in equilibrium with the surrounding air at height z and temperature T. It is then adiabatically displaced vertically by a displacement δz_0. Assume that throughout the motion, the air parcel always has the same pressure as the surrounding air at the same height. The surrounding atmosphere is unaltered and has a different lapse rate Γ. Neglect viscosity.

C.5	Find the equation of motion for δz, the instantaneous vertical displacement. Under what condition is the equilibrium at z stable? What is the angular frequency ω of small oscillation? Express your answers in terms of $T, \Gamma, g, \mu_{\text{air}}$ and c_p.	1.4 pt

Part D. Moisture (2.7 points)

Even though water constitutes a small portion of the atmosphere, it has a significant role in climate science. It is responsible for precipitation, and it is the most significant greenhouse gas. The phase of water depends on what temperature and pressure the water system is at, depicted on a $p - T$ phase diagram, see *Figure D.1*. When the pressure and temperature lie on the coexistence curve, both liquid and vapor water can be present in the system. The slope of the coexistence curve is given by the Clausius-Clapeyron equation:

$$\frac{dp_s}{dT} = \frac{\Delta S}{\Delta V}$$

where p_s is the saturation pressure, the pressure at the phase transition, ΔS and ΔV are the changes in entropy and volume across the phase transitions, respectively. Treat water vapor as an ideal gas.

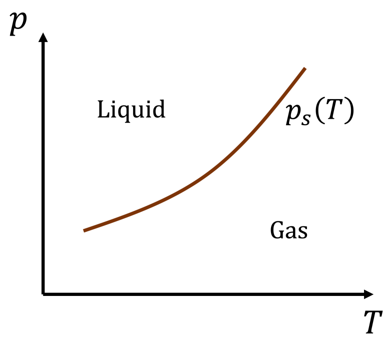

Figure D.1

D.1	Express dp_s/dT for the water liquid-vapor coexistence curve in terms of the water latent heat of evaporation $L, \mu_{\mathrm{H_2O}}, p_s, T$ and fundamental constants.	0.5 pt

D.2	If for some reference temperature T_o, $p_s = p_{so}$, find an expression for $p_s(T)$ in terms of $p_{so}, \mu_{\mathrm{H_2O}}, L, T, T_o$ and fundamental constants.	0.2 pt

Now we consider a `moist' air mass that rises adiabatically starting from a temperature T_i. The mass mixing ratio of water vapor (the mass of water vapor relative to the total mass) is ϕ. Take the air mass to have a specific molar heat at constant pressure c_p. The universal gas constant is $R = 8.31\ \mathrm{J/(mol\ K)}$.

D.3	Assuming that the air mass starts at $T_i = 17.0^\circ\mathrm{C}$ and $p_i = 10^5\ \mathrm{Pa}$. Find the temperature T_l at which liquid water starts forming in it if $\phi = 10^{-2}$. Assume that the water content in the air mass stays constant during the rise. Use $L = 2460\ \mathrm{kJ/kg}$ and $p_{so} = 1.94 \times 10^3\ \mathrm{Pa}$ at $T_i = 17.0^\circ\mathrm{C}$.	2.0 pt

Part E. Sun halo (1.6 points)

Under suitable atmospheric conditions, a bright ring appears around the Sun which is called a halo. Halos are caused by ice crystals present in the upper troposphere. One interesting feature about halos is that they always appear at a specific angle relative to the direction of the Sun.

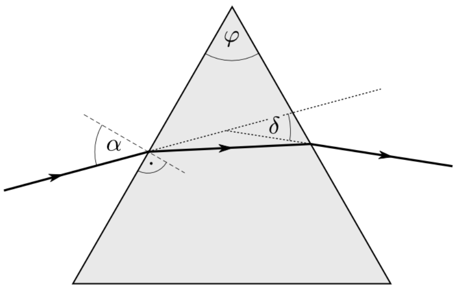

Figure E.1. On the left: A photograph showing a halo around the Sun. On the right: The path of a light ray passing through the prism.

E.1	Consider a simple prism with an apex angle of φ and direct a light ray onto it at an incidence angle α, as shown in *Figure E.1*. Let the refractive index of the prism be *n*. Express the angle of deviation δ of the light ray after passing through the prism in terms of α, n and φ.	0.8 pt

The most common type of halo forms when tiny ice crystals take the shape of regular hexagonal prisms. Light from the Sun falls onto randomly oriented ice crystals drifting in the atmosphere and scatters into various directions. However, in certain specific directions, the intensity of the refracted light is maximal, and this determines the angle at which the bright ring appears.

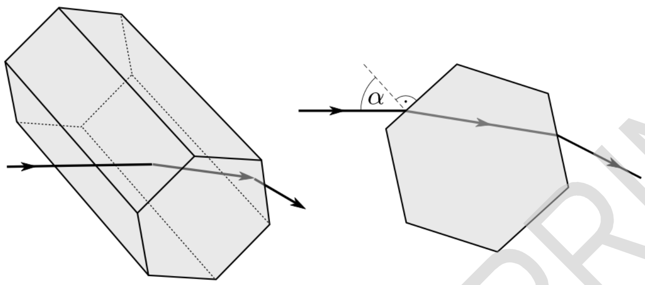

Figure E.2.

Consider a hexagonal ice prism whose six-fold symmetry axis is perpendicular to the direction of the Sun's rays. Investigate a light ray that refracts through two rectangular faces of the prism indicated in *Figure E.2*. Due to the random orientation of the ice crystals, the light strikes the crystal faces at varying incidence angles α.

E.2	Plot on the answer sheet how the deviation angle δ of the examined light ray depends on the incidence angle α within the interval $[20^\circ, 70^\circ]$ in 5° increments. The refractive index of ice is n=1.31.	0.6 pt

E.3	Using the graph from the previous part, determine at what angle the halo appears the brightest relative to the direction of the Sun.	0.2 pt